VOL. 18, NO. 12, DECEMBER 1980 AIAA JOURNAL 1485

ARTICLE NO. 80-1375R

Comparison of Methods for Prediction of Transition by Stability Analysis

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Several methods of transition prediction by linear stability analysis are compared. The spectral stability analysis code SALLY is used to analyze flows over laminar flow control wings. It is shown that transition prediction by the envelope method and a new modified wave packet method are comparable in reliability but that the envelope method is more efficient computationally.

Nomenclature

A = maximum disturbance amplitude
a_n = Chebyshev coefficients
c = wing chord
f = dimensional frequency
$\boldsymbol{k}$ = wave vector
L = algebraic mapping parameter
M = number of Chebyshev polynomials
N = N factor, $\ln A/A_0$
R = displacement thickness Reynolds number, $U_x\delta^*/\nu_\infty$
Re_c = chord Reynolds number, $U_\infty c/\nu_\infty$
s = arc length along an arbitrary path on the wing
T_n = Chebyshev polynomial
t = time
U = unperturbed x velocity in the boundary layer
$\boldsymbol{U}_p$ = potential flow vector at edge of boundary layer
U_x = x component of U_p
U_∞ = incoming freestream velocity
$\boldsymbol{v}_g$ = group velocity vector
v' = perturbation velocity in the y direction
W = unperturbed z velocity in the boundary layer
w = mapped coordinate normal to wing surface
x = coordinate in the direction of the normal chord
y = coordinate normal to the wing surface
z = coordinate along the wing span
α = x wavenumber
α_A = angle of attack
β = z wavenumber
ω = frequency
δ^* = displacement thickness
λ = wavelength
ν = kinematic viscosity
θ = wing sweep angle
ψ = angle formed by the wavenumber vector with the x axis
ψ_g = angle formed by the group velocity vector with the x axis
ψ_p = angle formed by the potential flow vector with the x axis
ϕ = eigenfunction, defined in Eq. (3)

Received Jan. 14, 1980; revision received June 23, 1980; presented as Paper 80-1375 at the AIAA 13th Fluid and Plasma Dynamics Conference, Snowmass, Colo., July 14-16, 1980.

Index categories: Boundary-Layer Stability and Transition; Computational Methods.

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Introduction

In this paper, several methods of transition prediction using linear stability analysis are compared. The incompressible linear stability computer code SALLY is used in various ways to study three-dimensional boundary-layer flow over laminar flow control (LFC) wings. Here we compare the so called envelope method[1] with wave-packet methods[2] to predict transition. We conclude that the envelope method is at least as reliable as the more complicated and less efficient wave-packet method.

Consider the stability of three-dimensional laminar flow over swept wings with sweep angle θ. The coordinate system used on the wing is depicted in Fig. 1. The x axis is in the direction of the normal chord, the y axis is normal to the surface of the wing, and the z axis is along its span.

Neglecting the curvature of the wing surface, compressibility effects, and nonparallel flow effects, linear disturbances satisfy the Orr-Sommerfeld equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2-\beta^2\right)^2\phi=iR\left\{(\alpha U+\beta W-\omega)\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2-\beta^2\right]\phi-\left(\alpha\frac{\mathrm{d}^2U}{\mathrm{d}y^2}+\beta\frac{\mathrm{d}^2W}{\mathrm{d}y^2}\right)\phi\right\} \tag{1}$$

with the boundary conditions

$$\phi(0)=\frac{\mathrm{d}\phi}{\mathrm{d}y}(0)=0; \qquad \phi(\infty)\text{ bounded} \tag{2}$$

Here the perturbation velocity in the y direction is assumed to be of the form

$$v'=Re\{\phi(y)\exp[i(\alpha x+\beta z-\omega t)]\} \tag{3}$$

$U(y)$ and $W(y)$ are the (unperturbed) laminar boundary-layer velocities in the x and z directions, respectively, and R is the Reynolds number. It is assumed that all variables are nondimensionalized with boundary-layer scaling.

Equations (1-3) constitute an eigenvalue problem for the frequency ω and wavenumbers α, β. For given Reynolds number R, this eigenvalue problem provides a complex dispersion relation of the form

$$\omega=\omega(\alpha,\beta) \tag{4}$$

relating the complex parameters α, β and ω.

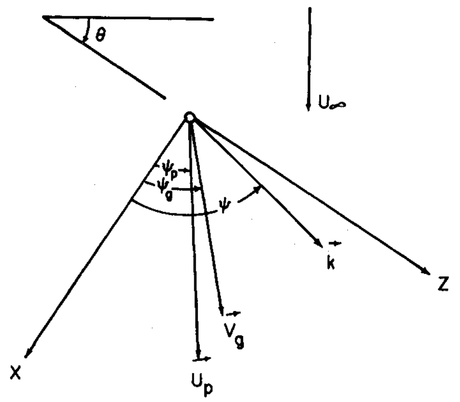

Fig. 1 A plot of the coordinate system on a swept wing.

Semiempirical methods to predict transition on LFC wings are based on tracing the evolution of modes across the wing.[1] An appropriate N factor for transition correlation is defined as the (logarithm of the) total growth factor across the wing. A good transition predictor is one for which transition occurs at nearly constant N for a wide variety of wings and flow conditions.

For natural transition, disturbances of all frequencies are present on the wing surface. In this case, there are many optional ways to compute N factors. The first choice is between temporal and spatial stability theory. In temporal theory, α and β are real while ω is complex; the mode grows in time if $\mathrm{Im}(\omega) > 0$, but the mode does not grow in space. An N factor for transition correlation may be defined as

$$N=\int_{s_0}^{s} \mathrm{Im}(\omega)/\mid \mathrm{Re}(v_g)\mid \mathrm{d}s \qquad (5)$$

where

$$v_g=(\partial\omega/\partial\alpha,\ \partial\omega/\partial\beta)$$

is the (complex) group velocity and s is the arclength along an appropriate curve on the wing. The N factor [Eq. (5)] is not fully defined until a prescription is given for singling out a specific mode at each position on the wing and for defining a specific curve on which to integrate. We shall return to these questions in the following section.

In spatial stability theory, ω is real but α and/or β may be complex. Again, there is arbitrariness in the definition of an appropriate N factor because of the variety of excitable modes on the wing.

Wave Propagation in Boundary Layers

The complex eigenvalue relation (4) provides two real relations among the three complex quantities α, β, and ω. In temporal stability theory, the requirements that α and β be real provide two more conditions so there remain two arbitrary parameters among $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta)$, $\mathrm{Re}(\omega)$, and $\mathrm{Im}(\omega)$.

There are several ways to remove this arbitrariness in the computation of the growth factors N. In the envelope method,[1] $\mathrm{Im}(\omega)$ is maximized with respect to α at fixed $\mathrm{Re}\ \omega$ (which then determines α, β and ω uniquely at each point on the wing) and the curve in Eq. (5) is defined to be everywhere tangent to $\mathrm{Re}(v_g)$.

With spatial stability theory, there remain three independent real parameters among α, β, and $\mathrm{Re}(\omega)$ once the eigenvalue condition Eq. (4) is satisfied. One possibility is to require that the direction of most rapid growth, which is parallel to the vector $[-\mathrm{Im}(\alpha),\ -\mathrm{Im}(\beta)]$, be parallel to $\mathrm{Re}(v_g)$ and that the resulting value of the most rapid growth rate be maximized with respect to the remaining two independent parameters.[3]

Alternatively, it is possible to use wave-packet theory to remove the arbitrariness in the definition of N factors. For a conservative dynamical system, kinematic wave theory implies that a wave packet propagates in physical and wavevector space according to the Hamilton-Jacobi equations[4]

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\partial\omega}{\partial\alpha} \qquad (6)$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial\omega}{\partial\beta} \qquad (7)$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t}=-\frac{\partial\omega}{\partial x} \qquad (8)$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}t}=-\frac{\partial\omega}{\partial z} \qquad (9)$$

Nayfeh[2] considered the extension of Eqs. (6-9) to nonconservative systems where α, β, and ω can be complex. Then, ω_α and ω_β may also be complex. For a physical solution with real x, z, and t to exist, Eqs. (6) and (7) imply that the group velocity $(\omega_\alpha, \omega_\beta)$ must be real. Nayfeh proposed the computation of wave-packet solutions determined by the independent conditions: 1) the eigenvalue condition (4); 2) $\mathrm{Im}\omega_\alpha=\mathrm{Im}\omega_\beta=0$; 3) $\mathrm{Re}\omega$ fixed; 4) $\mathrm{Re}\beta$ fixed; and 5) $\mathrm{d}x/\mathrm{d}t=\omega_\alpha$, $\mathrm{d}y/\mathrm{d}t=\omega_\beta$. Under these conditions the N factor is determined by

$$N=\int_{t_0}^{t}[-\omega_\alpha \mathrm{Im}(\alpha)-\omega_\beta \mathrm{Im}(\beta)+\mathrm{Im}(\omega)]\mathrm{d}t \qquad (10)$$

Finally, we study a modified nonconservative wave-packet formulation in which α, β, and ω are determined by: 1) the eigenvalue condition (4); 2) $\mathrm{Im}\ \omega_\alpha = \mathrm{Im}\ \omega_\beta = 0$;3) $\mathrm{Re}\ \omega$ fixed with $\mathrm{Im}\omega = 0$; and 4) $\mathrm{d}x/\mathrm{d}t = \omega_\alpha$, $\mathrm{d}y/\mathrm{d}t = \omega_\beta$. The motivation for these latter conditions is simply that laminar flow over a LFC wing may be assumed steady so a wave packet should propagate at fixed real frequency. However, there is less justification for assuming $\mathrm{Re}\beta$ is fixed as in Nayfeh's formulation, because the flow is not homogeneous in space. The N factor is given by Eq. (10) with $\mathrm{Im}(\omega) = 0$.

With Nayfeh's formulation of the wave packet equations, the growth factor N is a function of the independent variables $\mathrm{Re}\ \omega$ and $\mathrm{Re}\ \beta$, while N is a function of only $\mathrm{Re}\ \omega$ in our wave packet formulation. Therefore, maximization of N over all allowable packets is computationally more efficient with our formulation. We have performed computations (not reported in detail here) with Nayfeh's wave-packet formulation and have found the computations to be extremely sensitive, with realistic solutions satisfying the required constraints at some wing locations but not at others and the overall N factor at transition highly variable. We do not believe these latter effects originate in the numerical scheme. In any case, the conclusion of the present paper – that the envelope method is at least as reliable as the wave-packet method and that it is considerably more efficient – would not be changed by comparisons with results obtained by Nayfeh's wave packet formulation.

Numerical Method

In the computer code SALLY,[1] Eqs. (1) and (2) are solved using a spectral method based on Chebyshev polynomials.[5] The boundary-layer direction y, $0 \le y < \infty$, is mapped into the finite interval $-1 \le w < 1$ by the algebraic mapping

$$w=2\frac{y}{y+L}-1 \qquad (11)$$

and $\phi(y)$ is approximated as the finite Chebyshev polynomial series

$$\phi(y) = \sum_{n=0}^{N} a_n T_n(w) \tag{12}$$

The resulting algebraic eigenvalue problem is solved globally (if a guess for the eigenvalue is not available) by a generalized QR algorithm or locally (if a good guess is available) by inverse Rayleigh iteration.[6] The resulting scheme is very efficient and accurate.

The properties of the laminar boundary-layer profiles required to solve Eqs. (1) and (2) are obtained using a compressible boundary-layer code for swept tapered wings developed by Kaups and Cebeci.[1]

The code SALLY also performs a number of optional computations, including: 1) computation of maximum amplification among all wavelengths and propagation angles; 2) computation of amplification at fixed frequency and fixed wavelength; 3) computation of amplification at fixed frequency and fixed propagation angle; 4) computation of maximum amplification at fixed frequency; and 5) computation of wave-packet solutions according to the prescriptions discussed in the previous section.

Results

Burrows[7] has reported flight transition data taken at Cranfield for a large, untapered, 45 deg swept half-wing mounted as a dorsal fin upon the midupper fuselage of an Avro Lancaster airplane. The airfoil section was made up of two semiellipses, one of which constituted a faired trailing edge and the other corresponding to the leading edge of a 10% thick airfoil, with effective chord of 10.83 ft, measured in the freestream direction. The location of the beginning of transition in the Cranfield data was estimated as given in Ref. 8. Two of the Cranfield flight results were chosen for correlating transition using wave-packet theory.

In the first test case, calculations were made for a chord Reynolds number of 11.7×10^6 and -2 deg angle of attack. In this flow, transition begins at $x/c = 5.5\%$. A maximum N factor of 7.6 was obtained at a frequency of 1250 Hz both

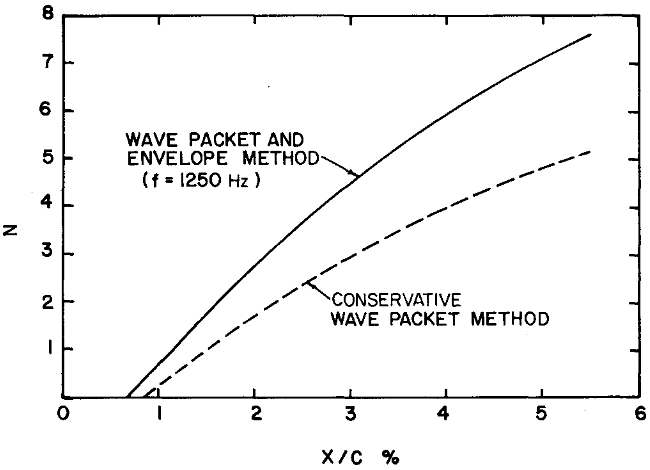

Fig. 2 A plot of N vs percent of chord x/c for various methods applied to a swept wing of an Avro Lancaster airplane at -2 deg angle of attack; solid curve: modified wave-packet method and envelope method at $f = 1250$ Hz which gives nearly the maximum N at the transition point; dashed curve: result of integrating Eqs. (6-9) across the wing with Eqs. (6) and (7) replaced by their real parts. The curves are plotted from the beginning of the unstable flow region until the transition point at $x/c = 5.5\%$.

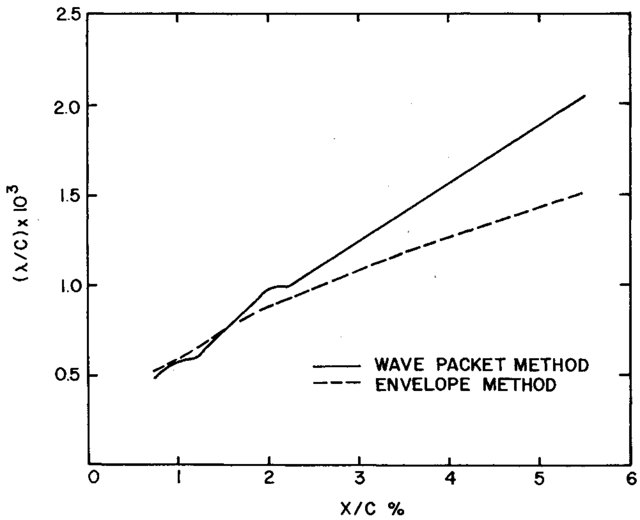

Fig. 4 A plot of wavelength vs x/c for the same flow as in Fig. 2.

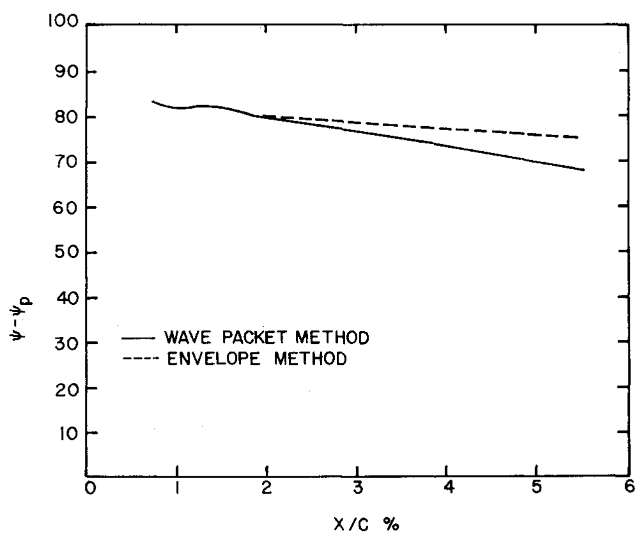

Fig. 3 A plot of wave propagation angle vs x/c for the same flow as in Fig. 2.

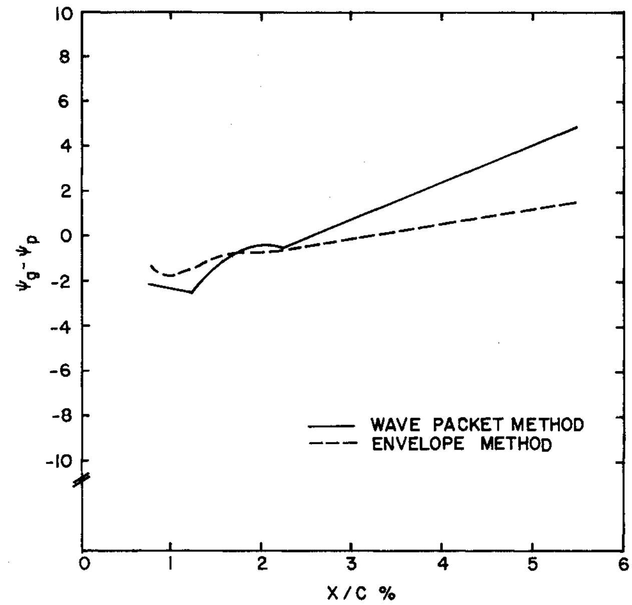

Fig. 5 A plot of the direction of the group velocity for the same flow as in Fig. 2.

with the envelope method and the modified wave-packet method.

The predicted variation of the N factor up to the transition location was almost identical for the envelope method and the modified wave-packet method. We also compute the solution of the conservative wave-packet Eqs. (6-9) in which only the real parts of Eqs. (6) and (7) are taken while Eqs. (8) and (9) are solved in their full complex form. The resulting N factor at transition is 5.2. The variation of N factor with x/c for the various methods is plotted in Fig. 2.

Wave angle, wavelength, and the direction of the group velocity as predicted by the envelope and wave-packet methods are given in Figs. 3-5. Although the results are qualitatively similar, there is appreciable quantitative difference in these parameters at the transition location. It is surprising that the N factor calculated by the envelope and modified wave-packet methods are the same.

In the second test case, the angle of attack of the wing was changed to zero. In this case, transition occurred experimentally at $x/c = 7\%$. The envelope method gave an N factor of 10.8 at a frequency of 1000 Hz. The wave-packet method gave a maximum N factor of 10.5 at a frequency of 1200 Hz, which is close to the prediction of the envelope method. The variation of N factor with x/c is plotted in Fig. 6. The predictions of the conservative wave-packet approximation and a fixed wavelength, fixed frequency integration are also plotted in this figure. The conservative wave-packet approximation gave an N factor at transition of 8.6 rather than 10.5.

Figure 7 shows the influence of frequency on N factor at transition for the wing as predicted by the wave-packet theory. Wave angle, wavelength and direction of the group velocity for this particular wing are shown in Figs. 8-10. Again there is substantial quantitative difference in the predictions of the two methods.

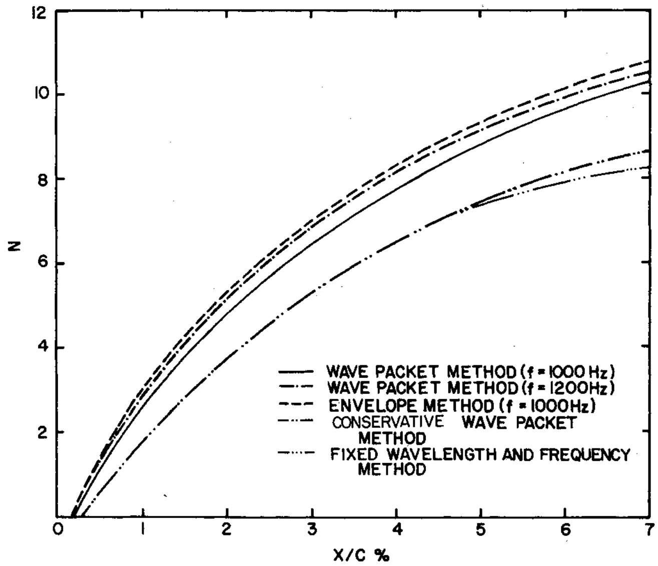

Fig. 6 Same as Fig. 2 except for the wing at 0 deg angle of attack. In addition to the results of the wave-packet methods and envelope method, the N factor obtained by integrating a fixed wavelength, fixed frequency mode across the wing is given. Here N is given by Eq. (5) and the mode is determined by the six real conditions: 1) Eq. (4); 2) $\text{Im}\alpha = \text{Im}\beta = 0$; 3) $\lambda/c = 0.001$; and 4) $\text{Re}\,\omega = 750$ Hz.

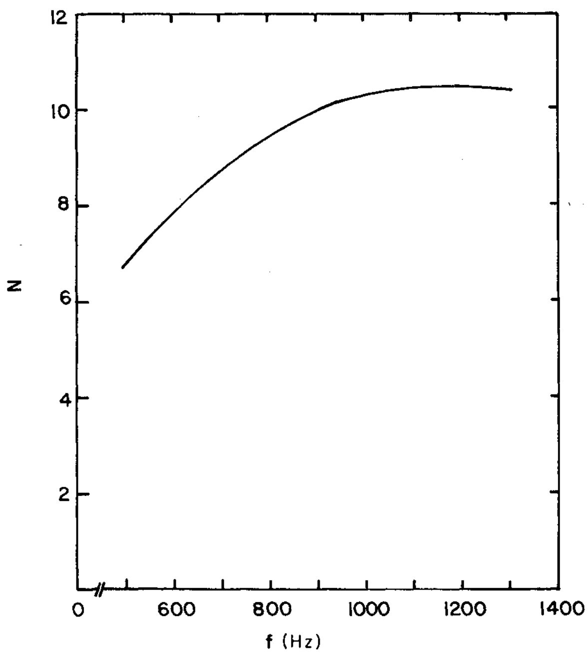

Fig. 7 Variation of N at transition vs frequency obtained using the modified wave-packet method for the same flow as in Fig. 6.

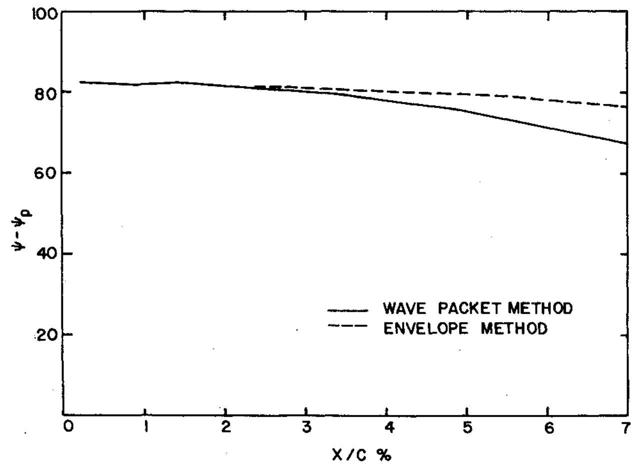

Fig. 8 A plot of wave propagation angle vs x/c for the same flow as in Fig. 6.

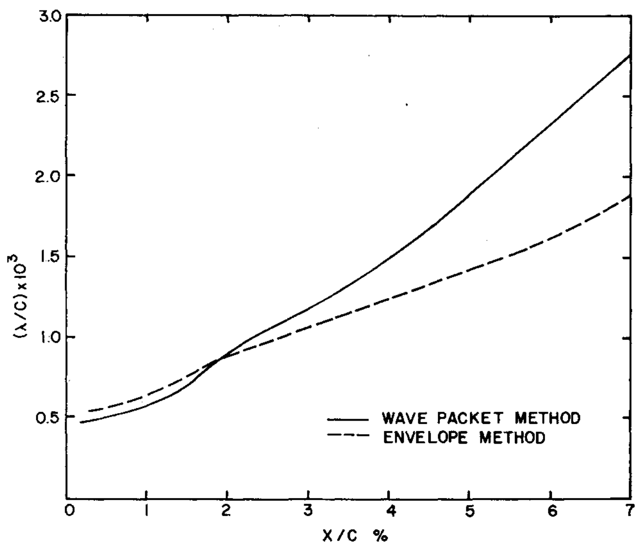

Fig. 9 A plot of wavelength vs x/c for the same flow as in Fig. 6.

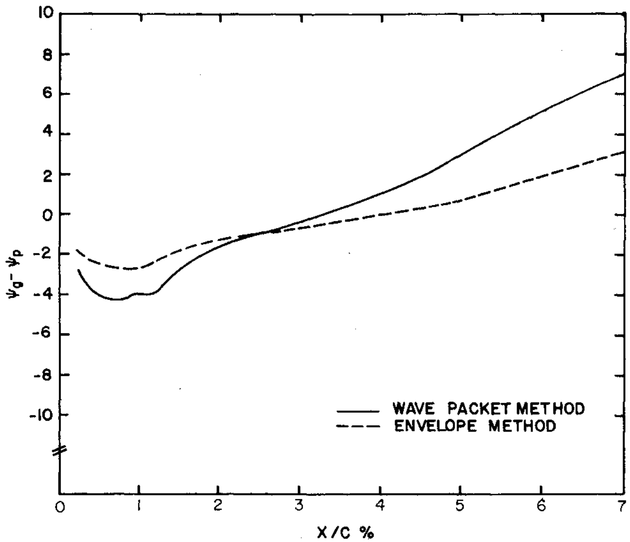

Fig. 10 A plot of the direction of the group velocity for the same flow as in Fig. 6.

Conclusions

Calculations were made for a Cranfield 45 deg swept wing with $Re_c = 11.7 \times 10^6$ using a modified wave-packet method and the envelope method. Both methods gave an N factor of 7.6 at transition location for an angle of attack, $\alpha_A = -2$ deg. For $\alpha_A = 0$ deg, the envelope and modified wave-packet methods gave N factors of 10.8 and 10.5, respectively. Since it may be argued that the wave-packet method is physically more relevant for predicting transition in three-dimensional boundary layers, it was initially hoped that the wave-packet method might give more consistent transition N factors. However, the results show that the wave-packet method provides N factors that are at best as consistent as those of envelope method. Since the wave-packet method is at least three times as expensive to use as the envelope method, the latter is recommended for engineering design calculations.

Acknowledgments

We would like to thank D. M. Bushnell and J. N. Hefner for helpful discussions. This work was supported by NASA under Contracts NAS1-15604 (MRM) and NAS1-15894 (SAO).

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